

# INFINITE DIMENSIONAL MANIFOLDS FROM A NEW POINT OF VIEW

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**ABSTRACT.** In this paper we propose a new treatment about infinite dimensional manifolds, using the language of categories and functors. Our definition of infinite dimensional manifolds is a natural generalization of finite dimensional manifolds in the sense that de Rham cohomology and singular cohomology can be naturally defined and the basic properties (Functorial Property, Homotopy Invariant, Mayer-Vietoris Sequence) are preserved. In this setting we define the classifying space  $BG$  of a Lie group  $G$  as an infinite dimensional manifold. Using simplicial homotopy theory and the Chern-Weil theory for principal  $G$ -bundles we show that de Rham's theorem holds for  $BG$  when  $G$  is compact. Finally we get, as an unexpected byproduct, two simplicial set models for the classifying spaces of compact Lie groups; they are totally different from the classical models constructed by Milnor, Milgram, Segal and Steenrod.

**Keywords:** infinite dimensional manifolds, de Rham theorem, classifying space of Lie group

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## 1. INTRODUCTION

Usually, infinite dimensional manifolds are defined to be paracompact spaces  $X$  modelled on some topological vector space  $E$ ;  $E$  may be a Fréche space, a Banach space or a Hilbert space. More explicitly,  $X$  is a paracompact space covered by an atlas of open subsets  $\{U_\alpha\}$  each of which is homeomorphic to

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an open set  $E_\alpha$  of  $E$  by a given homeomorphism  $\phi_\alpha : U_\alpha \rightarrow E_\alpha$ . The transitive functions between charts are assumed to be infinitely differentiable. The meaning of infinitely differentiable is too complicated to be given here; we refer the reader to [7, 20] for details. In this setting differential forms can be defined and de Rham's theorem holds under some mild conditions.

In this paper we propose a new treatment about infinite dimensional manifolds. Our treatment differs from the usual one in three respects. First, we adopt the language of categories and functors and our definition of infinite dimensional manifolds is a natural generalization of finite dimensional manifolds in the sense that de Rham cohomology and singular cohomology can be naturally defined and the basic properties (Functorial Property, Homotopy Invariant, Mayer-Vietoris Sequence) are preserved. Second, we need no topology except the topologies of finite dimensional manifolds. Thus we would not talk about the topology of an infinite dimensional manifold; instead we still have the homotopy type of an infinite dimensional manifold. Finally we can naturally define many classes of infinite dimensional manifolds including spaces of smooth mappings between finite dimensional manifolds, groups of diffeomorphisms of finite dimensional manifolds, spaces of connections on principal  $G$ -bundles and classifying spaces of Lie groups.

In §2 we give the definition of infinite dimensional manifolds and several examples. §3 is concerned with the cohomology theory of infinite dimensional manifolds. In §4 we prove that de Rham's theorem holds for the classifying spaces of compact Lie groups. During the proof we get, as an unexpected byproduct, two simplicial set models for the classifying spaces of compact Lie groups; they are totally different from the classical models constructed by Milnor, Milgram, Segal and Steenrod (cf.[17, 19, 23, 24]). Throughout this paper, we assume that the reader is familiar with the elementary simplicial homotopy theory and category theory. The standard references are [4, 6, 14].

## 2. DEFINITIONS AND EXAMPLES

Let  $\mathbb{R}_k^n \subset \mathbb{R}^n$  be the subspace

$$\mathbb{R}_k^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, 1 \leq i \leq k\}.$$

A function  $f$  on an open subset  $U \subset \mathbb{R}_k^n$  is *smooth* if and only if  $f$  can be extended to a  $C^\infty$  function on an open subset of  $\mathbb{R}^n$ . A map  $g$  from an open subset  $U \subset \mathbb{R}_k^n$  to an open subset  $U' \subset \mathbb{R}_l^m$  is *smooth* if  $f_i \circ g$  is smooth for each coordinate function  $f_i$ ,  $1 \leq i \leq m$ .

**Definition 2.1.** A chart (with corners) on a topological space  $X$  is a homeomorphism  $\phi : U \rightarrow X_\alpha$  of an open subset  $U$  of  $\mathbb{R}_k^n$  with an open subset  $X_\alpha$  of  $X$ . Two charts  $\phi_1, \phi_2$  are said to be compatible if the functions  $\phi_2^{-1}\phi_1$  and  $\phi_1^{-1}\phi_2$  are smooth.

**Definition 2.2.** A smooth atlas (with corners) on a space  $X$  is an countable open cover  $\{X_\alpha\}$  of  $X$  together with a collection of compatible charts  $\phi_\alpha : U_\alpha \rightarrow X_\alpha$ . A smooth manifold with corners is a paracompact Hausdorff space  $X$  together with a maximal smooth atlas, i.e., a smooth structure.

As usual we can define *smooth map* and *smooth homeomorphism* between smooth manifolds with corner. We also have the concepts of closed submanifold and open submanifold, and the Whitney embedding theorem still holds in this setting; the proof is routine. From now on, we assume that all smooth manifolds are Hausdorff, paracompact and have a countable base.

Denote by  $\mathcal{K}$  the category of smooth manifolds with corner and smooth maps. For any category  $\mathcal{L}$ , let  $ob(\mathcal{L})$  be the class of objects of  $\mathcal{L}$ . We say that  $\mathcal{L}$  is a *small category* if  $ob(\mathcal{L})$  forms a set.

**Theorem 2.3.** There exists a subcategory  $\mathcal{M}$  of  $\mathcal{K}$  satisfying:

- (0)  $\mathcal{M}$  is a small category;
- (1) the inclusion functor  $i : \mathcal{M} \rightarrow \mathcal{K}$  is an equivalence of categories, i.e.,  $i$  is full and for any object  $M$  in  $\mathcal{K}$ , there is object  $N$  in  $\mathcal{M}$  such that  $i(N)$  is isomorphic to  $M$ ;
- (2)  $\mathbb{R}^n$  belongs to  $\mathcal{M}$  for each  $n \geq 0$ ;
- (3) let  $N$  be a (closed or open) submanifold of  $M \in ob(\mathcal{M})$ , then  $N \in ob(\mathcal{M})$ ;
- (4)  $ob(\mathcal{M})$  is closed under finite product;
- (5) let  $M_1, M_2, \dots, M_n, \dots$  be an infinite sequence of objects of  $\mathcal{M}$ , and for each  $0 < k < \infty$  let

$$f_k : M_k \longrightarrow N$$

be an isomorphism of  $M_k$  onto a (closed or open) submanifold of  $N \in ob(\mathcal{K})$ , satisfying  $f_k|_{M_j \cap M_k} = f_j|_{M_j \cap M_k}$  for each  $j, k$ , then there is an  $N' \in ob(\mathcal{M})$ , including each  $M_k$  as a (closed or open) submanifold, and an isomorphism  $j : N' \longrightarrow N$  such that the composition  $M_k \hookrightarrow N' \rightarrow N$  is  $f_k$ .

*Proof.* We want to choose, by transfinite induction, for each at most countable ordinal  $\alpha$  a full subcategory  $\mathcal{M}_\alpha$  of  $\mathcal{K}$ . First let  $\mathcal{M}_0$  be the full subcategory of closed submanifolds of  $\mathbb{R}^n$ . As each  $M \in ob(\mathcal{K})$  can be

closely embedded in an  $\mathbb{R}^n$ , we see that  $\mathcal{M}_0$  satisfies conditions (0), (1) and (2). Let  $\beta$  be a fixed at most countable ordinal. Suppose for all ordinals  $\alpha$  with  $\alpha < \beta$ , we have chosen  $\mathcal{M}_\alpha$ ; all of them are small categories. If  $\beta$  is a limit ordinal, set  $\mathcal{M}_\beta = \bigcup_{\alpha < \beta} \mathcal{M}_\alpha$ . If  $\beta$  is a successor ordinal, i.e.,  $\beta = \gamma + 1$  for some  $\gamma$ , we choose  $\mathcal{M}_\beta$  to be a full subcategory of  $\mathcal{K}$  (such a choice can always be done by adding some new objects) satisfying:

- (0')  $\mathcal{M}_\beta$  is a small category;
- (3') let  $N$  be a (closed or open) submanifold of  $M \in ob(\mathcal{M}_\gamma)$ , then  $N \in ob(\mathcal{M}_\beta)$ ;
- (4') for any  $M_1, M_2, \dots, M_n \in ob(\mathcal{M}_\gamma)$ ,  $M_1 \times M_2 \times \dots \times M_n$  belongs to  $\mathcal{M}_\beta$ ;
- (5') let  $M_1, M_2, \dots, M_n, \dots$  be an infinite sequence of objects of  $\mathcal{M}_\gamma$ , and for each  $0 < k < \infty$  let

$$f_k : M_k \longrightarrow N$$

be an isomorphism of  $M_k$  onto a (closed or open) submanifold of  $N \in ob(\mathcal{K})$  satisfying  $f_k|_{M_k \cap M_j} = f_j|_{M_k \cap M_j}$ , then there exists an  $N' \in ob(\mathcal{M}_\beta)$ , including each  $M_k$  as a (closed or open) submanifold, and an isomorphism  $j : N' \longrightarrow N$  such that the composition  $M_k \hookrightarrow N' \rightarrow N$  is  $f_k$ .

Now set  $\mathcal{M} = \bigcup_\alpha \mathcal{M}_\alpha$  where  $\alpha$  runs over all at most countable ordinals. It is easy to see from our construction that  $\mathcal{M}$  satisfies conditions (0)-(5).  $\square$

Let  $p : P \rightarrow M$  be a principal  $G$ -bundle in  $\mathcal{M}$ , i.e.,  $P, M \in ob(\mathcal{M})$  (from now on, all principal  $G$ -bundles are assumed to be in  $\mathcal{M}$ ), and let  $f : N \rightarrow M$  be an arrow in  $\mathcal{M}$ . Then the pullback of  $p : P \rightarrow M$  along  $f$  is also a principal  $G$ -bundle in  $\mathcal{M}$ . Similar statement holds for vector bundles in  $\mathcal{M}$ .

**Definition 2.4.** Let  $(Set)$  be the category of sets. A smooth functor is a contravariant functor

$$F : \mathcal{M}^{op} \rightarrow (Set).$$

A smooth functor  $F$  is called separated (resp. an infinite dimensional manifold) if it satisfies the following condition. Assume that  $\bigcup_i V_i$  is an open covering of  $X$  in  $\mathcal{M}$ , and for each  $i$  there is an  $\alpha_i \in F(V_i)$  such that  $\alpha_i|_{V_i \cap V_j} = \alpha_j|_{V_i \cap V_j}$  for each  $i, j$ , then there is at most (resp. exactly) one  $\alpha \in F(X)$  such that  $\alpha|_{V_i} = \alpha_i$ .

**Remark 2.5.** Using the language of sheaf theory (cf. [1, 15, 18]), a smooth functor is just a presheaf on the site of smooth manifolds, and a separated smooth functor (resp. an infinite dimensional manifold) is a separated presheaf (resp. a sheaf).

**Definition 2.6.** Let  $F, G$  be smooth functors, a smooth map from  $F$  to  $G$  is a natural transformation from  $F$  to  $G$ . Denote by  $\widetilde{\mathcal{M}}$  the category of smooth functors and smooth maps, and denote by  $\widetilde{\mathcal{M}}'$  and  $\mathcal{N}$  the full subcategories of separated smooth functors and infinite dimensional manifolds respectively.

**Lemma 2.7.** The inclusion functor  $i : \widetilde{\mathcal{M}}' \hookrightarrow \widetilde{\mathcal{M}}$  has a left adjoint functor  $\varsigma : \widetilde{\mathcal{M}} \rightarrow \widetilde{\mathcal{M}}'$ , and the inclusion functor  $i' : \mathcal{N} \hookrightarrow \widetilde{\mathcal{M}}'$  has a left adjoint functor  $\varsigma' : \widetilde{\mathcal{M}}' \rightarrow \mathcal{N}$

This lemma is standard in sheaf theory (cf.[15, p.128]), we give a proof for later use.

*Proof.* Given a smooth functor  $F$ , we construct  $\varsigma(F)$  as follows. Let  $M$  be an object in  $\mathcal{M}$ , we say that  $\alpha, \beta \in F(M)$  are equivalent if there is an open covering  $\bigcup_i M_i$  of  $M$  such that  $\alpha|_{M_i} = \beta|_{M_i}$  for each  $i$ . Define  $\varsigma(F)(M)$  to be the equivalent classes of elements of  $F(M)$ . It is clear that  $\varsigma(F)(M)$  is functorial in  $M$  and  $\varsigma(F)$  is a separated smooth functor.

Given a separated smooth functor  $G$ , we construct  $\varsigma'(G)$  as follows. Let  $M$  be an object in  $\mathcal{M}$ , a *local datum* of  $G$  on  $M$  consists of an open covering  $\bigcup_i M_i$  of  $M$ , and for each  $i$  an  $\alpha_i \in G(M_i)$ , satisfying  $\alpha_i|_{M_i \cap M_j} = \alpha_j|_{M_i \cap M_j}$  for any  $i, j$ . Two local data  $(\bigcup_i M_i, \alpha_i)$  and  $(\bigcup_j N_j, \beta_j)$  of  $M$  on  $G$  are said to be *equivalent* if  $\alpha_i|_{M_i \cap N_j} = \beta_j|_{M_i \cap N_j}$  for any  $i, j$ . Define  $\varsigma'(G)(M)$  to be the set of equivalent classes of local data of  $G$  on  $M$ . Then  $\varsigma'(G)(M)$  is functorial in  $M$  and  $\varsigma'(G)$  is an infinite dimensional manifold. One checks that  $\varsigma$  and  $\varsigma'$  are adjoint functors of  $i$  and  $i'$  respectively.  $\square$

**Definition 2.8.** Let  $F$  be an infinite dimensional manifold, a submanifold of  $F$  is an infinite dimensional manifold  $G$  such that, for each  $M$  in  $\mathcal{M}$ ,  $G(M)$  is a subset of  $F(M)$ .

Given two submanifolds  $G, H$  of an infinite dimensional manifold  $F$ , define  $G \sqcup H$  in  $\widetilde{\mathcal{M}}$  by  $G \sqcup H(M) = G(M) \cup H(M)$ . Set  $G \cup H = \varsigma'(G \sqcup H)$ , then  $G \cup H$  is also a submanifold of  $F$ , called the *union* of  $G$  and  $H$ . The *intersection* of  $G$  and  $H$  is the submanifold  $G \cap H$  defined by  $G \cap H(M) = G(M) \cap H(M)$ . The *product* of two infinite dimensional manifolds  $F, G$  is the infinite dimensional manifold  $F \times G$  defined by  $F \times G(M) = F(M) \times G(M)$ .

To each  $M$  in  $\mathcal{M}$ , we can associate a contravariant functor

$$h_M : \mathcal{M}^{op} \rightarrow (Set)$$

which sends  $N$  in  $\mathcal{M}$  to the set  $Hom_{\mathcal{M}}(N, M)$ ; if  $\alpha : N \rightarrow N'$  is an arrow in  $\mathcal{M}$ , then  $h_M(\alpha)$  is defined by composition with  $\alpha$ . It is clear that  $h_M$

is an infinite dimensional manifold. An arrow  $f : M \rightarrow N$  in  $\mathcal{M}$  yields a smooth map, i.e., a natural transformation  $h_f : h_M \rightarrow h_N$ . In this setting, the Yoneda Lemma states that,

**Lemma 2.9.** *Let  $M$  and  $N$  be objects in  $\mathcal{M}$ , then the function*

$$\text{Hom}_{\mathcal{M}}(M, N) \rightarrow \text{Hom}_{\mathcal{N}}(h_M, h_N)$$

*that sends  $f : M \rightarrow N$  to  $h_f$  is bijective.*

From now on we always identify each  $M$  in  $\mathcal{M}$  with the infinite dimensional manifold  $h_M$  in  $\mathcal{N}$ . Thus  $\mathcal{M}$  is a full subcategory of  $\mathcal{N}$ .

**Example 2.10.** *Given  $M, N$  in  $\mathcal{M}$ , we can associate an infinite dimensional manifold  $C^\infty(M, N)$  by sending  $L$  in  $\mathcal{M}$  to  $\text{Hom}_{\mathcal{M}}(L \times M, N)$ , if  $f : L \rightarrow L'$  is an arrow in  $\mathcal{M}$ ,  $C^\infty(M, N)(f)$  is defined by composition with  $f \times \mathbb{I} : L \times M \rightarrow L' \times M$ .*

**Example 2.11.** *For any  $M$  in  $\mathcal{M}$ , define  $\text{Diff}(M)$  in  $\mathcal{N}$  by sending  $N$  in  $\mathcal{M}$  to the set*

$$\{f \in \text{Hom}_{\mathcal{M}}(N \times M, M) \mid f|_{\{x\} \times M} \text{ is a diffeomorphism for each } x \in N\}.$$

*For an arrow  $f : L \rightarrow L'$  in  $\mathcal{M}$ ,  $\text{Diff}(M)(f)$  can be defined analogously.*

**Example 2.12.** *Let  $\xi = \{E \rightarrow M\}$  in  $\mathcal{M}$  be a vector bundle or a principal  $G$ -bundle, define  $\text{Sec}(\xi)$  in  $\mathcal{N}$  by sending each  $N$  in  $\mathcal{M}$  to the set of smooth sections of  $P^*\xi = \{P^*E \rightarrow M \times N\}$  where  $P : M \times N \rightarrow M$  is the projection. For each arrow  $f : N \rightarrow N'$ ,  $\text{Sec}(\xi)(f)$  is defined by the pullback of sections along  $\text{id} \times f : M \times N \rightarrow M \times N'$ .*

**Example 2.13.** *Let  $G$  be a Lie group, define  $BG$  in  $\mathcal{N}$  by sending each  $M$  in  $\mathcal{M}$  to the set of principal  $G$  bundles on  $M$  with connection; for an arrow  $f : M \rightarrow N$  in  $\mathcal{M}$ ,  $BG(f)$  is defined by the pullback of principal  $G$  bundles and connections along  $f$ . From the condition (5) in Theorem 2.3 we see that  $BG$  is indeed an infinite dimensional manifold. Our definition of  $BG$  is suggested by the existence of universal connection on the classifying space of Lie group (cf.[21, 22]).*

**Remark 2.14.** *As pointed out by the referee, this example has an analogy in algebraic geometry, where the classifying space of an algebraic group  $G$  is defined as a stack which classifies  $G$ -torsors (cf.[13, p.11]), but  $G$ -torsors are nothing but principal  $G$ -bundles in algebraic geometry.*

**Example 2.15.** *Let  $\xi = \{E \rightarrow M\}$  be a principal  $G$ -bundle, the space of connections on  $\xi$  is an infinite dimensional manifold  $\text{Con}(\xi)$  defined as follows. For each object  $N$  in  $\mathcal{M}$ ,  $\text{Con}(\xi)(N)$  is the set of connections on the principal  $G$ -bundle  $P^*(\xi)$ , where  $P : M \times N \rightarrow M$  is the projection.*

For an arrow  $f : N \rightarrow N'$  in  $\mathcal{M}$ ,  $\text{Con}(\xi)(f)$  is defined by the pullback of connections along  $\text{id} \times f : M \times N \rightarrow M \times N'$ .

**Example 2.16.** For each  $n \geq 0$ ,  $\mathcal{A}^n \in \mathcal{N}$  is defined by sending each  $M$  in  $\mathcal{M}$  to the set of differential  $n$ -forms on  $M$ ; for each arrow  $f : M \rightarrow N$  in  $\mathcal{M}$ ,  $\mathcal{A}^n(f)$  is the induced map  $f^* : \mathcal{A}^n(N) \rightarrow \mathcal{A}^n(M)$ . The exterior derivative induces a smooth map  $d^n : \mathcal{A}^n \rightarrow \mathcal{A}^{n+1}$ , and the exterior product induces a smooth map  $\wedge : \mathcal{A}^m \times \mathcal{A}^n \rightarrow \mathcal{A}^{m+n}$ .

### 3. COHOMOLOGY THEORY

Let  $F$  be a smooth functor, a differential  $n$ -form on  $F$  is a smooth map from  $F$  to  $\mathcal{A}^n$ . Denote by  $\mathcal{A}^n(F)$  the real vector space of differential  $n$ -forms on  $F$ , the smooth map  $d^n : \mathcal{A}^n \rightarrow \mathcal{A}^{n+1}$  induces a differential operator  $d^n(F) : \mathcal{A}^n(F) \rightarrow \mathcal{A}^{n+1}(F)$ . The complex  $\mathcal{A}^*(F)$  together with the differential operators  $d^*(F)$  is called the *de Rham complex* on  $F$ . Applying the smooth map  $\wedge : \mathcal{A}^m \times \mathcal{A}^n \rightarrow \mathcal{A}^{m+n}$  we see that the  $\mathcal{A}^*(F)$  has the structure of a commutative differential graded algebra, the corresponding cohomology ring  $H_{de}^*(F)$  is called the *de Rham cohomology* of  $F$ .

**Definition 3.1.** Let  $F$  be a smooth functor, the singular complex  $S(F)$  is the simplicial set given by

$$S(F)_n = F(|\Delta^n|) = \text{Hom}_{\widetilde{\mathcal{M}}}(|\Delta^n|, F),$$

where

$$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1, t_i \geq 0\}$$

is the geometric realization of the standard  $n$ -simplicial set  $\Delta^n$ . The singular cohomology ring  $H_{si}^*(F; R)$  of  $F$  with coefficient ring  $R$  is the cohomology ring  $H^*(S(F); R)$ ; we can analogously define singular (co)chain complex and the homology (homotopy) groups of  $F$ .

Thus, although we have no *topology* for an infinite dimensional manifold, we can still talk about the homotopy type of it. In the following two propositions  $H^*(F)$  will be  $H_{de}^*(F)$  or  $H_{si}^*(F; R)$ .

**Proposition 3.2. (Functorial Property)** Each smooth map  $f : F \rightarrow G$  in  $\widetilde{\mathcal{M}}$  induces a ring homomorphism  $f^* : H^*(G) \rightarrow H^*(F)$ . If  $g : G \rightarrow L$  is another smooth map in  $\widetilde{\mathcal{M}}$ , then  $(gf)^* = f^*g^*$ .

Let  $\mathbb{I} \in \mathcal{M}$  be the unit interval  $[0, 1]$  and let  $f, g : F \rightarrow G$  be two smooth maps in  $\widetilde{\mathcal{M}}$ . We say that  $f$  is *smooth homotopic* to  $g$  if there is a smooth map  $h : F \times \mathbb{I} \rightarrow G$  such that  $h(\cdot, 0) = f$  and  $h(\cdot, 1) = g$ .

**Proposition 3.3. (Homotopy Invariant)** *Let  $f, g : F \rightarrow G$  be two smooth maps in  $\widetilde{\mathcal{M}}$ , if  $f$  is smooth homotopic to  $g$ , then  $f^* = g^* : H^*(G) \rightarrow H^*(F)$ .*

The proofs of these two proposition are routine, we omit the details.

**Proposition 3.4.** *For any  $F$  in  $\mathcal{M}$  and  $G$  in  $\mathcal{M}'$ , the canonical maps  $h : F \rightarrow \varsigma(F)$  and  $h' : G \rightarrow \varsigma'(G)$  induce isomorphisms on singular cohomology and de Rham cohomology.*

*Proof.* We give the proof only in the case of singular cohomology. First we show that  $h$  and  $h'$  induce isomorphisms on singular homology. Let  $C_*(F)$  be the singular chain complex of  $F$  and let  $S$  be the subdivision operator  $S(F) : C_*(F) \rightarrow C_*(F)$ , it was shown in [8, p.121] that  $S(F)$  is chain homotopic to the identity map. Let  $H'_*(F)$  be the homology groups of the complex  $\ker h_*$ , where  $\ker h_*$  is the kernel of the surjective homomorphism  $h_* : C_*(F) \rightarrow C_*(\varsigma(F))$ . As  $S$  induces isomorphisms on  $H_*(F)$  and  $H_*(\varsigma(F))$ , it also induces isomorphisms on  $H'_*(F)$ . On the other hand, by the definition of  $\varsigma(F)$  in the proof of Lemma 2.7, we see that for any singular chain  $\alpha \in \ker h_*$  we have  $S^n(\alpha) = 0$  for sufficiently large  $n$ . Hence we have  $H'_*(F) \cong 0$  and  $h$  induces isomorphism on singular homology.

Now we show that for any separated smooth functor  $G$  the canonical map  $h' : G \rightarrow \varsigma'(G)$  induces isomorphism on singular homology. We only need to repeat the argument of the previous paragraph except that the kernel of  $h_*$  should be replaced by the cokernel of  $h'_*$ , the details are omitted. By the universal coefficients theorem the same is true for cohomology.  $\square$

**Proposition 3.5. (Mayer-Vietoris Sequence)** *Let  $F$  be an infinite dimensional manifold and let  $G, H$  be two submanifolds of  $F$  such that  $G \cup H = F$ , then we have the following long exact sequence*

$$\begin{aligned} \cdots \longrightarrow H_{si}^{n-1}(G \cap H; R) &\longrightarrow H_{si}^n(F; R) \longrightarrow H_{si}^n(G; R) \oplus H_{si}^n(H; R) \\ &\longrightarrow H_{si}^n(G \cap H; R) \longrightarrow H_{si}^{n+1}(F; R) \longrightarrow \cdots \end{aligned}$$

*Proof.* This proposition follows directly from Proposition 3.4 and the definition of  $G \cup H$ .  $\square$



**Remark 3.6.** For any topological space  $X$ , we can define an infinite dimensional manifold  $h_X$  as follows. For each object  $N$  in  $\mathcal{M}$ ,  $h_X(N)$  is the set of all continue maps from  $N$  to  $X$ . For an arrow  $f : N \rightarrow N'$  in  $\mathcal{M}$ ,  $h_X(f)$  is defined by composition with  $f$ . Let  $A \cup B$  be an open covering of  $X$ , then  $h_A$  and  $h_B$  are submanifolds of  $h_X$  with  $h_A \cup h_B = h_X$ . Applying the above proposition to  $(h_X, h_A, h_B)$ , we get the usual Mayer-Vietoris Sequence for the triple  $(X; A, B)$ .

**Remark 3.7.** It is pointed out by the referee that a smooth functor  $F$  yields a simplicial presheaf by

$$M \mapsto F(\Delta^\bullet \times M),$$

and the above cohomology theory follows directly from the homotopy theory of simplicial presheaf [9, 10].

Let  $F$  be an infinite dimensional manifold. To each differential  $n$ -form  $\omega : F \rightarrow \mathcal{A}^n$  on  $F$ , we will associate a singular  $n$ -cochain  $\tilde{\omega}$  on  $F$  as follow. For each  $\alpha \in F(\Delta^n)$ ,  $\omega(\alpha)$  is a differential  $n$ -form (in the usual sense) on  $\Delta^n$ ; set  $\tilde{\omega}(\alpha) = \int_{\Delta^n} \omega(\alpha)$ . It is clear that this assignment  $\omega \rightarrow \tilde{\omega}$  induces a homomorphism from the de Rham complex to the singular cochain complex with real coefficient, denote by  $\mathfrak{R}(F) : H_{de}^*(F) \rightarrow H_{si}^*(F; \mathbb{R})$  the induced homomorphism of cohomology. The famous de Rham theorem states that  $\mathfrak{R}(M)$  is an isomorphism for each  $M$  in  $\mathcal{M}$ . In the general case we don't know under what conditions will  $\mathfrak{R}(F)$  be an isomorphism. In fact, a crucial step in the finite dimensional case is the Poincaré Lemma (cf.[2]), but we cannot prove an analogous lemma for infinite dimensional manifolds. We refer the reader to [16] for the proof of de Rham theorem of infinite dimensional manifolds in the usual sense.

**Remark 3.8.** Note that in this section all the definitions, results and their proofs coincide with the usual ones in the finite dimensional case except the definition of singular (co)homology. When  $F \in \widetilde{\mathcal{M}}$  is represented by  $M$  in  $\mathcal{M}$ ,  $H_{si}^*(F; \mathbb{R})$  is not the singular cohomology of  $M$  but the cohomology defined by smooth singular (co)chains. The equivalence of these two cohomologies was proved in [5].

#### 4. CLASSIFYING SPACES OF LIE GROUPS

In this section we show that the de Rham theorem is valid for  $BG$  in  $\mathcal{N}$  where  $G$  is a compact Lie group in  $\mathcal{M}$ .

Consider  $BG' \in \widetilde{\mathcal{M}}$  which sends each  $M$  in  $\mathcal{M}$  to the set of principal  $G$ -bundles on  $M$ . The map  $p' : BG \rightarrow BG'$  is defined by neglecting the connections.

**Theorem 4.1.** *The geometric realization  $|S(BG)|$  of the simplicial set  $S(BG)$  is a classifying space of  $G$ .*

*Proof.* We divide the proof into four lemmas.

**Lemma 4.2.**  *$S(BG')$  is a connected fibrant.*

*Proof.* From condition (5) of Theorem 2.3, we see that  $S(BG')$  is a connected simplicial set. Thus it remains to show that given a commutative diagram in the category of simplicial set  $\mathbb{S}$  as follow (where  $\Delta^n$  is the standard  $n$ -simplicial set and  $\Lambda_k^n$  is the  $k$ -th horn of  $\Delta^n$ ):

$$(1) \quad \begin{array}{ccc} \Lambda_k^n & \longrightarrow & S(BG') \\ i \downarrow & \nearrow \theta & \downarrow p \\ \Delta^n & \longrightarrow & *, \end{array}$$

there is a map  $\theta : \Delta^n \rightarrow S(BG')$  (the dotted arrow) making the diagram commute. Equivalently given a principal  $G$ -bundle on  $|\Lambda_k^n|$  we want to extend it to a principal  $G$ -bundle on  $|\Delta^n|$ . As the inclusion  $|\Lambda_k^n| \hookrightarrow |\Delta^n|$  is a homotopy equivalence, such an extension always exists by condition (5) of Theorem 2.3.  $\square$

**Lemma 4.3.**  *$|S(BG')|$  is a classifying space of  $G$ .*

*Proof.* Let  $\mathbb{B}G$  (a  $CW$  complex) be a classifying space of  $G$ . For any finite subcomplex  $K$  of  $|S(BG')|$ , there is a canonical principal  $G$ -bundle  $\xi_K$  over  $K$  with projection  $E_K \rightarrow K$  (glue together principal  $G$ -bundles on all simplices), hence a classifying map  $l_K : K \rightarrow \mathbb{B}G$ . As  $\{l_K\}$  are compatible (up to homotopy) under inclusions of finite subcomplexes of  $|S(BG')|$ , by the homotopy extension property of  $CW$  complexes (cf.[8]), they induce a map  $l : |S(BG')| \rightarrow \mathbb{B}G$  such that the restriction of  $l$  to  $K$  is homotopic to  $l_K$  for each finite subcomplex  $K$ . In order to prove this lemma, it suffices to show that  $l$  induces isomorphisms  $l_*$  on homotopy groups.

First we prove that  $l_*$  is surjective. Fix a faithful representation  $i : G \rightarrow U(n)$  (for the existence of such a representation, see [3], p.136), and consider the principal  $G$ -bundle  $\delta_m$  with projection  $U(m+n)/U(m) \rightarrow U(m+n)/(U(m) \times G)$  for each  $m$ . Choose a triangulation of  $U(m+n)/(U(m) \times G)$ , then it induces a classifying map (by the definition of  $BG'$ )  $k : U(m+n)$

$n)/(U(m) \times G) \rightarrow |S(BG')|$  such that the composition  $l \cdot k$  is the classifying map for  $\delta_m$ . As  $U(m+n)/U(m)$  is  $2m$ -connected, the classifying map  $l \cdot k$  induces isomorphism on  $\pi_i$  for  $i < 2m$  (cf.[25], p.202), hence  $l_*$  is surjective on homotopy groups.

Now we prove that  $l_*$  is injective. Let  $\alpha \in \pi_n(|S(BG')|)$  satisfying  $l_*(\alpha) = 0$ , we want to show that  $\alpha = 0$ . As  $S(BG')$  is a fibrant, we can represent  $\alpha$  by a simplicial map  $\alpha : \Delta^n \rightarrow S(BG')$  such that  $\alpha|_{\partial\Delta^n}$  is the constant map  $\partial\Delta^n \rightarrow * \hookrightarrow S(BG')$  (where  $*$  is represented by the trivial  $G$ -bundle  $G \rightarrow *$ ).

By the definition of  $BG'$ ,  $\alpha$  is represented by a principle  $G$ -bundle  $\xi$  on  $|\Delta^n|$  satisfying  $\xi|_{\partial\Delta^n}$  is trivial, i.e. a principle  $G$ -bundle  $\xi$  on  $|\Delta^n/\partial\Delta^n| \cong S^n$ . Now  $l_*(\alpha) = 0$  implies  $\xi$  is isomorphic to the trivial  $G$ -bundle

$$G \times |\Delta^n/\partial\Delta^n| \longrightarrow |\Delta^n/\partial\Delta^n|.$$

Now condition (5) of Theorem 2.3 implies that there is a principal  $G$ -bundle  $\delta$  over  $|\Delta^n| \times \mathbb{I}$  together with a trivialization of  $\delta|_{(|\partial\Delta^n| \times \mathbb{I})}$ , such that  $\delta|_{(|\Delta^n| \times \{0\})} = \xi$  and  $\delta|_{(|\Delta^n| \times \{1\})}$  is the trivial  $G$ -bundle. This yields a homotopy from  $\alpha$  to the constant map, thus  $\alpha = 0$ .  $\square$

**Lemma 4.4.** *Let  $\mathbb{R}_+^n = \{(t_1, \dots, t_n) \in \mathbb{R}^n | t_i \geq 0\}$  and  $H_i = \{(t_1, \dots, t_n) \in \mathbb{R}_+^n | t_i = 0\}$ . Given a smooth function  $f_i$  on  $H_i$  for each  $i$ , satisfying  $f_i|_{H_i \cap H_j} = f_j|_{H_i \cap H_j}$ , there always exists a smooth function  $f$  on  $\mathbb{R}_+^n$  such that  $f|_{H_i} = f_i$ .*

*Proof.* Extend  $f_n$  to a smooth function  $f'_n$  on  $\mathbb{R}_+^n$  by

$$f'_n(t_1, \dots, t_n) = f_n(t_1, \dots, t_{n-1}, 0).$$

If we can find a smooth function  $f''$  satisfying  $f''|_{H_i} = f_i - f'_n$ , then  $f'' + f'_n$  is the desired  $f$ . Thus it suffices to prove this lemma when  $f_n \equiv 0$ . Repeating this argument we see that it suffices to prove this lemma when  $f_i = 0$  for any  $i \leq n$ ; in this case the lemma is trivial.  $\square$

**Lemma 4.5.** *The induced simplicial map  $p : S(BG) \rightarrow S(BG')$  is a fibration.*

*Proof.* It suffices to prove that for every commutative diagram in  $\mathbb{S}$

$$(2) \quad \begin{array}{ccc} \Lambda_k^n & \longrightarrow & S(BG) \\ i \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & S(BG') \end{array}$$

there is a map  $\theta : \Delta^n \rightarrow S(BG)$  (the dotted arrow) making the diagram commute.

By the definition of  $BG$  and  $BG'$ , the above diagram gives a trivial principle  $G$ -bundle  $\xi$  on  $|\Delta^n|$  and for each  $i \neq k$  a connection  $w_i$  on  $(d^i)^*(\xi)$ , satisfying  $w_i = w_j$  when restrict to intersection of two faces. To find a map  $\theta : \Delta^n \rightarrow S(BG)$  making the diagram commute is equivalent to find a connection  $w$  on  $\xi$  such that when restricted to each  $(d^i)^*(\xi)$  ( $i \neq k$ ),  $w = w_i$ . But in a trivial  $G$ -bundle, a connection is just a  $\mathfrak{g}$ -value differential 1-form on the base space, where  $\mathfrak{g}$  is the Lie algebra of  $G$ . By a linear coordinate transformation and Lemma 4.4 we see that there always exists such a connection on  $\xi$ .  $\square$

**Lemma 4.6.** *The fibre  $f^{-1}(*)$  is contractible.*

*Proof.* By Lemma 5.1. in [6, p.190], it suffices to show that  $f^{-1}(*)$  is connected and has an extra degeneracy. The the set of  $n$ -simplices of  $f^{-1}(*)$  is the set of connections on the trivial principal  $G$ -bundle  $p : G \times |\Delta^n| \rightarrow |\Delta^n|$ , i.e., the set of  $\mathfrak{g}$ -value differential 1-forms on  $|\Delta^n|$ . Thus there is only one 0-simplex in  $f^{-1}(*)$ . To each  $\mathfrak{g}$ -value differential 1-form  $w$  on  $|\Delta^n|$  we assign a  $\mathfrak{g}$ -value differential 1-form  $s_{-1}(w)$  on  $|\Delta^{n+1}|$  as follows.

Set

$$\Delta^+ = \{(t_0, \dots, t_{n+1}) \in |\Delta^{n+1}| \mid t_0 \leq \frac{1}{2}\}$$

and

$$\Delta^- = \{(t_0, \dots, t_{n+1}) \in |\Delta^{n+1}| \mid t_0 \geq \frac{1}{2}\}.$$

Then  $|\Delta^{n+1}| = \Delta^+ \cup \Delta^-$ . Define a projection  $f : \Delta^+ \rightarrow |\Delta^n|$  by

$$f(t_0, \dots, t_{n+1}) = (\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0})$$

and set

$$s_{-1}(w)|_{\Delta^+} = (e^4 e^{-(\frac{1}{2}-t_0)^{-2}}) f^*(w), \quad s_{-1}(w)|_{\Delta^-} = 0.$$

One checks that  $s_{-1}(w)$  is a  $\mathfrak{g}$ -value differential 1-form on  $|\Delta^{n+1}|$  and this assignment gives an extra degeneracy on  $f^{-1}(*)$ . Thus the proof is done.  $\square$

Applying Lemma 4.3, Lemma 4.5, Lemma 4.6 and long exact sequence of homotopy groups in [6, p.28] we see that  $p' : S(BG) \rightarrow S(BG')$  induces isomorphisms on homotopy groups, hence  $|S(BG)|$  is also a classifying space of  $G$ . This completes the proof of Theorem 4.1.  $\square$

**Proposition 4.7.**  $\mathfrak{R}(BG) : H_{de}^*(BG) \rightarrow H_{si}^*(BG; \mathbb{R})$  is an isomorphism. In other words, de Rham's theorem holds for  $BG$ .

*Proof.* By the definition of  $BG$ , the differential forms on  $BG$  are gauge natural differential forms. But Theorem 52.8 in [12, p.403] states that all gauge natural differential forms are the classical Chern-Weil forms. Hence from Theorem 4.1 and the Chern-Weil theory (we refer the readers to [11, Ch.12] for a detailed exposition about Chern-Weil theory) we see that

$$\mathfrak{R}(BG) : H_{de}^*(BG) \rightarrow H_{si}^*(BG; \mathbb{R})$$

is an isomorphism.  $\square$

**Remark 4.8.** *We see that  $S(BG)$  and  $S(BG')$  are two new (simplicial set) models for the classifying spaces of compact Lie groups; they are totally different from the classical models constructed by Milnor, Milgram, Segal and Steenrod [17, 19, 23, 24].*

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